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The importance of a module in a system has been a useful concept in Reliability Theory. Several definitions of this concept are available. One such has been called the role of a module in the failure of a system. It is measured by the probability that the module has failed at the time of the failure of the system. In an earlier paper we studied interesting properties of this measure for a class of second order c-out-of-d systems.

In this paper we consider the role of a group of modules. As before this is measured by the probability that at least s modules from this group of modules have failed at the time of the failure of the system, for $s = 1, 2, \ldots$. We use the tools of arrangement increasing functions and majorization to study monotonicity properties of this measure in terms of the parameters of the system.

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Abstract

The importance of a module in a system has been a useful concept in Reliability Theory. Several definitions of this concept are available. One such has been called the role of a module in the failure of a system. It is measured by the probability that the module has failed at the time of the failure of the system. In an earlier paper we studied interesting properties of this measure for a class of second order c-out-of-d systems.

In this paper we consider the role of a group of modules. As before this is measured by the probability that at least s modules from this group of modules have failed at the time of the failure of the system, for $s = 1, 2, \ldots$. We use the tools of arrangement increasing functions and majorization to study monotonicity properties of this measure in terms of the parameters of the system.

1. Introduction

The importance of a component in a system is a useful concept in Reliability Theory, The importance of a component may be measured in many ways. It may be measured by the increment in reliability of the system per unit increase in the reliability of the component. This view is taken in the pioneering paper of Birnbaum (1969). Boland, El-Neweihi and Proschan (1988) and Natvig (1985) have built upon this concept of importance.

The probability that a component is among the components that failed before the failure of a system provides another measure of the importance of the component. This view can be found in Fussell and Vesely (1972) and Barlow and Proschan (1975).

A general summary of many different ways to measure the importance of a component may be found in the expository paper of Boland and El-Neweihi (1990).

A system generally consists of modules which themselves are subsystems of individual components. In El-Neweihi and Sethuraman (1991) we defined the role of a module to be the probability that it is among the modules that have failed at the time of the failure of the system.

In our earlier work El-Neweihi, Proschan and Sethuraman (1978) and El-Neweihi and Sethuraman (1991) we studied the role of a module in the failure of some special systems which are subclasses of general second order r-out-of-k systems. A second order r-out-of-k system is defined as follows. Let $P_1, P_2, \ldots, P_k$ be k modules with no common components where each module is a a_i-out-of-n_i system. The system S fails as soon as $k - r + 1$ of the modules $P_1, P_2, \ldots, P_k$ fail. In El-Neweihi, Proschan and Sethuraman (1978) we considered a series-parallel system and studied the role of a particular parallel subsystem. In El-Neweihi and Sethuraman (1991) we allowed the modules $P_1, P_2, \ldots, P_k$ to be parallel systems and assumed that the system S was a $k - r + 1$-out-of-k system based on these modules and studied the role of a particular module. In that paper we also allowed the module P_i to be an a_i-out-of-n_i system, $i = 1, \ldots, k$ and took the system S to be series system based on these modules and once again studied the role of a particular module. We also considered structures which were dual to the above structures. The monotonicity properties of the role of a module as a function of the parameters of the system were derived and applications to optimal allocation were given in that paper.

In this paper we study the role of a group of modules. There can be several ways to define the role of a group of modules just as there can be several ways to define the role of a module. In this paper we define the role of a group of modules to be the probability that at least s of the modules in this group are among the modules that have failed at the time of the failure of the system, for $s = 1, 2, \ldots$.

We will not give here the definitions and properties of arrangement increasing functions and majorization; these can be found for instance in Marshall and Olkin (1979). For two distribution functions F and G, we will say that $F \leq G$ if $F(t) \leq G(t)$ for all t. Using this ordering and the theory of arrangement increasing functions and majorization, we study the monotonicity properties of the role of a group of modules in terms of the parameters of the system in Sections 2 and 3. Some of these monotonicity properties are extensions of the results in El-Neweihi, Proschan and Sethuraman (1978) and El-Neweihi and Sethuraman (1991) , while others are of a different nature arising from the fact that

we are considering the role of a group of modules rather than the role of a single module.

All the above results can be generalized to second order c-out-of-d multistate systems. As an illustration, we give in Section 4, a typical generalization of such monotonicity properties to some special classes of multistate systems.

The monotonicity properties of this paper have applications to optimal allocation along the lines of similar applications in El-Neweihi and Sethuraman (1991) and are therefore not described in this paper.

2. General second order r-out-of-k systems and the role of a group of modules

Consider a collection of $l+k$ modules $Q_1, \ldots, Q_l$, and $P_1, \ldots, P_k$. We assume that Q_i contains m_i components whose lifetimes have a common continuous distribution $G_i(x)$, $i = 1, \ldots, l$, and that P_j contains n_j components whose lifetimes have a common continuous distribution $F_j(x)$, $j = 1, \ldots, k$. We also assume that all the $m_1 + \cdots + m_l + n_1 + \cdots + n_k$ components are independent. Let $\mathbf{m}$ denote $(m_1, \ldots, m_l)$ and let $\mathbf{n}$ denote $(n_1, \ldots, n_k)$. When $m_1 = \ldots = m_l = m$, we let m stand for $\mathbf{m}$ and when $n_1 = \ldots = n_k = n$, we let n stand for $\mathbf{n}$. Again, let $\mathbf{G}$ denote $(G_1, \ldots, G_l)$. When $G_1 = \ldots = G_l = G$, we let G stand for $\mathbf{G}$. Similarly we let $\mathbf{F}$ denote $(F_1, \ldots, F_k)$ and once again when $F_1 = \ldots = F_k = F$, we let F stand for $\mathbf{F}$.

Throughout this paper we will assume that Q_i is a $b_i + 1$-out-of-m_i system, $i = 1, \ldots, l$ and that P_j is a $a_j + 1$-out-of-n_j system, $j = 1, \ldots, k$. Let $\mathbf{b}$ denote $(b_1, b_2, \ldots, b_l)$ and $\mathbf{a}$ denote $(a_1, a_2, \ldots, a_k)$. Again when $b_1 = b_2 = \cdots = b_l = b$ we denote $\mathbf{b}$ by b. Similarly when $a_1 = a_2 = \cdots = a_k = a$ we denote $\mathbf{a}$ by a.

Let $\{X_{iq}, q = 1, \ldots, m_i\}$ be the lifetimes of the components in module Q_i, $i = 1, \ldots, l$ and let $\{Y_{jp}, p = 1, \ldots, n_j\}$ be the lifetimes of the components in module P_j, $j = 1, \ldots, k$. Let X_i be the lifetime of the module Q_i, $i = 1, \ldots, l$ and let Y_j be the lifetime of the module P_j, $j = 1, \ldots, k$. Let $M_1, \ldots, M_l, N_1, \ldots, N_k$ be the ranks of the lifetimes $X_1, \ldots, X_l, Y_1, \ldots, Y_k$. Let $X_{(1)}, \ldots, X_{(l)}$ be the order statistics of $X_1, \ldots, X_l$, $Y_{(1)}, \ldots, Y_{(k)}$ be the order statistics of $Y_1, \ldots, Y_k$. Let $Z_{(1)}, \ldots, Z_{(l+k)}$ be the order statistics of $X_1, \ldots, X_l, Y_1, \ldots, Y_k$.

Let $h_{c|d}(p_1, \ldots, p_d) = P(\sum_{i=1}^{d} U_i \geq c)$ where $U_1, \ldots, U_d$ are independent Bernoulli random variables with $P(U_i = 1) = p_i, i = 1, \ldots, d$. The function $h_{c|d}(p_1, \ldots, p_d)$ is

the reliability function of a c-out-of-d system with component reliabilities $p_i, i = 1, \ldots, d$. When $p_1 = p_2 = \cdots = p_d = p$, we denote the reliability function $h_{c|d}(p_1, \ldots, p_d)$ by $h_{c|d}(p)$.

The distribution of $X_{(s)}$ can be easily written in terms of this reliability function as follows.

$$P(X_{(s)} > t) = h_{(l-s+1)|l}(h_{(b_1+1)|m_1}(\bar{G}_1(t)), \ldots, h_{(b_l+1)|m_l}(\bar{G}_l(t))).$$

One can write down the distribution of $Y_{(r)}$ in a similar fashion.

Let $P\{(s,r); (l, \mathbf{b}, \mathbf{m}, \mathbf{G}); (k, \mathbf{a}, \mathbf{n}, \mathbf{F})\}$ denote $P(X_{(s)} < Y_{(r)}) = P(X_{(s)} \leq Z_{(r+s-1)})$. By conditioning on $Y_{(r)}$, we get the following.

$$\begin{aligned} P(X_{(s)} < Y_{(r)}) &= \int P(X_{(s)} < t) dF_{Y_{(r)}}(t) \\ &= 1 - \int P(X_{(s)} > t) dF_{Y_{(r)}}(t) \\ &= 1 - \int h_{(l-s+1)|l}(h_{(b_1+1)|m_1}(\bar{G}_1(t)), \ldots, h_{(b_l+1)|m_l}(\bar{G}_l(t))) dF_{Y_{(r)}}(t) \quad (2.1) \end{aligned}$$

where $F_{Y_{(r)}}(t)$ is the distribution function of $Y_{(r)}$.

By conditioning on $X_{(s)}$ we can get the following alternative expression for the above.

$$\begin{aligned} P(X_{(s)} < Y_{(r)}) &= \int P(Y_{(r)} > t) dF_{X_{(s)}}(t) \\ &= \int h_{(k-r+1)|k}(h_{(a_1+1)|n_1}(\bar{F}_1(t)), \ldots, h_{(a_k+1)|n_k}(\bar{F}_k(t))) dF_{X_{(s)}}(t) \quad (2.2) \end{aligned}$$

where $F_{X_{(s)}}(t)$ is the distribution function of $X_{(s)}$.

We will now give two interpretations to the probability $P(X_{(s)} < Y_{(r)})$.

1 Consider a system S which is a $(l+k) - (s+r-1) + 1$-out-of-$(l+k)$ system based on the modules $Q_1, \ldots, Q_l$, and $P_1, \ldots, P_k$. The above probability is the probability that at least s of the modules from the group $Q_1, \ldots, Q_l$ have failed at the time of the failure of the system. This measures the role of the group of modules $Q_1, \ldots, Q_l$ in the failure of the system S.

2 Suppose that we have a subsystem S_1 which is a $(l-s+1)$-out-of-l system based on the modules $Q_1, \ldots, Q_l$ and a subsystem S_2 which is a $(k-r+1)$-out-of-k system based on the modules $P_1, \ldots, P_k$. Let S be a series system based on S_1 and S_2.

The above probability is then the probability that the failure of the system S is due to the failure of the subsystem S_1. This probability can be viewed as the role of S_1 in the failure of the system S, a third order 2-out-of-2 system.

Theorem 2.1 below lists several monotonicity properties of the function $P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$.

Theorem 2.1

The function $P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$

1 *is nonincreasing in* s *and nondecreasing in* r *and*

2 *is nonincreasing in* l *and nondecreasing in* k.

3 $P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$ *is decreasing in* $\mathbf{m},\mathbf{F},\mathbf{a}$ *and increasing in* $\mathbf{G},\mathbf{b},\mathbf{n}$.

4 $P\{(s+1,r);(l+1,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\} \geq P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$.

5 $P\{(s,r+1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k+1,\mathbf{a},\mathbf{n},\mathbf{F})\} \leq P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$.

Proof. The proofs of (1)-(3) above follow from the simple observation that $P(X_{(s)} \leq Y_{(r)})$ increases when $X_{(s)}$ decreases stochastically or $Y_{(r)}$ increases stochastically, and that $X(s)$ and $Y_{(r)}$ are stochastically increasing in $X_1,\ldots,X_l,Y_1,\ldots,Y_k$. The proofs of (4)-(5) above follow from the the observation that $h_{(c+1)|(d+1)}(p_1,\ldots,p_{(d+1)}) \leq h_{(c)|(d)}(p_1,\ldots,p_{(d)})$ since $P(\sum_{i=1}^{d+1} U_i \geq c+1) = P(\sum_{i=1}^{d} U_i + U_{d+1} \geq c+1) \leq P(\sum_{i=1}^{d} U_i + 1 \geq c+1) = P(\sum_{i=1}^{d} U_i \geq c)$. $\diamondsuit$

In the next section we study more involved inequalities based on majorization and arrangement increasing functions when we specialize the parameters of the system.

3. Further monotonicity properties for the role of a group of modules

In this section, we will establish more involved monotonicity properties of $P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$ by specializing the parameters of the system. These are given in Theorems 3.1–Theorem 3.7 below.

For our next theorem we assume that $a_i = 0, F_i = F, i = 1,\ldots,k$, *i.e*, the modules $P_1,\ldots,P_k$ are parallel subsystems with common lifetime distributions for their components.

Theorem 3.1 $P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,0,\mathbf{n},F)\}$ *is Schur concave in* $\mathbf{n}$.

Proof. Recall that

$$P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,0,\mathbf{n},F)\} = \int P(Y_{(r)} > t)dF_{X_{(s)}}(t).$$

Notice that

$$P(Y_{(r)} > t) = h_{(k-r+1)|k}(h_{1|n_1}(\bar{F}_l(t)),\ldots,h_{1|n_k}(\bar{F}_k(t)))$$

which by the Theorem 2.2 of Pledger and Proschan (1971) is a Schur-concave function of $\mathbf{n}$. This proves Theorem 3.1. $\diamondsuit$

For the next theorem let $a_i = 0, n_i = n, i = 1,\ldots,k$ and $\bar{F}_i(t) = \exp(-\lambda_i R(t)), i = 1,\ldots,k$ and $r = 1$. This is usually referred to as the proportional hazard case. Then $P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$ is a function that depends on $\mathbf{F}$ through $\boldsymbol{\lambda}$ and in Theorem 3,2 below, we will denote it by $P^*\{(s,1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,0,n,\boldsymbol{\lambda})\}$. Theorem 3.2 follows immediately from Theorem 2.6 of El-Neweihi and Sethuraman (1991) .

Theorem 3.2 *The function* $P^*\{(s,1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,0,n,\boldsymbol{\lambda})\}$ *is Schur-concave in* $\boldsymbol{\lambda}$.

In the following theorem let $a_i = 0, n_i = n, i = 1,\ldots,k$ and $F_i(t) = \exp(-\lambda_i A(t)), i = 1,\ldots,k$. This is usually referred to as the proportional left-hazard case. In this case, notice that $P\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},\mathbf{n},\mathbf{F})\}$ is a function that depends on $\mathbf{F}$ through $\boldsymbol{\lambda}$ which we will denote by $P_*\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,0,n,\boldsymbol{\lambda})\}$ in Theorem 3.3 below which follows immediately from Theorem 2.7 of El-Neweihi and Sethuraman (1991) .

Theorem 3.3 *The function* $P_*\{(s,r);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,0,n,\boldsymbol{\lambda})\}$ *is Schur-concave in* $\boldsymbol{\lambda}$.

For the next theorem let $a_i = a, i = 1,\ldots,k$ and $r = 1$, *i.e.* the module P_i is an $a+1$-out-of-n_i subsystem, $i = 1,\ldots,k$. In this case the system S fails if s of the modules $Q_1,\ldots,Q_l$, and $P_1,\ldots,P_k$ fail. The probability that s of the modules in $Q_1,\ldots,Q_l$ fail before any module in $P_1,\ldots,P_k$ is given by $P\{(s,1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,a,\mathbf{n},\mathbf{F})\}$. Theorem 3.3 below follows from Theorem 3.2 of El-Neweihi and Sethuraman (1991) .

Theorem 3.4 $P\{(s,1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,a,\mathbf{n},\mathbf{F})\}$ *is AI in* $(\mathbf{n},\mathbf{F})$.

In the next theorem we assume that $n_i = n, F_i = F, i = 1,\ldots,k$ and $r = 1$. In this case the modules $P_1,\ldots,P_k$ have the same number of components all with common lifetime distributions and S fails if s of the modules $Q_1,\ldots,Q_l$, and $P_1,\ldots,P_k$ fail. Theorem 3.5 below follows from Theorem 3.4 of El-Neweihi and Sethuraman (1991) .

Theorem 3.5 $P\{(s,1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,\mathbf{a},n,F)\}$ *is Schur-concave in* $\mathbf{a}$.

Suppose that $l + r - 1$ is fixed in advance at u. Let N_u be the number of modules from $Q_1, \ldots, Q_l$ that fail at the time of the failure of the system. We will now obtain some monotonicity properties of $E(N_u)$ in terms of the parameters of the system. Notice that

$$\begin{aligned} E(N_u) &= \sum_{s=1}^{u} P(N_u \geq s) \\ &= \sum_{s=1}^{u} P\{(s, u-s+1); (l, \mathbf{b}, \mathbf{m}, \mathbf{G}), (k, \mathbf{a}, \mathbf{n}, \mathbf{F})\}. \end{aligned}$$

Theorems 3.1-3.5 of this section described monotonicity properties of the summand in the above. These monotonicity properties will also be therefore inherited by $E(N_u)$.

In the above theorems we can reverse the role of $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_l$ and obtain the reverse inequalities in terms of the parameters of the modules $Q_1, \ldots, Q_l$.

In Theorems 3.6 and 3.7 below we assume that $s = r = 1$. Then the structure S is a series structure based on the modules $Q_1, \ldots, Q_l$, and $P_1, \ldots, P_k$. The probability $P(X_{(1)} < Y_{(1)})$ is the probability that the failure of system S is caused by the failure of one of the modules $Q_1, \ldots, Q_l$.

Assume further that $b_j = b, a_i = a, j = 1, \ldots, l, i = 1, \ldots, k$. Let $m_1 \leq m_2 \leq \cdots \leq m_l$ and $n_1 \leq n_2 \leq \cdots \leq n_k$. Suppose that $\mathbf{G}'$ is better arranged than $\mathbf{G}$ and $\mathbf{F}$ is better arranged than $\mathbf{F}'$. The following theorem compares the systems when we change the lifetime distributions from $\mathbf{G}, \mathbf{F}$ to $\mathbf{G}', \mathbf{F}'$.

Theorem 3.6

$$P\{(1,1); (l, b, \mathbf{m}, \mathbf{G}), (k, a, \mathbf{n}, \mathbf{F})\} \geq P\{(1,1); (l, b, \mathbf{m}, \mathbf{G}'), (k, a, \mathbf{n}, \mathbf{F}')\}.$$

Proof. This theorem follows from the fact that as we switch from $\mathbf{G}, \mathbf{F}$ to $\mathbf{G}', \mathbf{F}'$ we decrease $Y_{(1)}$ stochastically and increase $X_{(1)}$ stochastically. $\diamondsuit$

Assume that $r = s = 1$ as before. Suppose that $m_1 = \cdots = m_l = m, G_1 = \cdots = G_L = G, n_1 = \cdots = n_k = n$, and $F_1 = \cdots, = F_k = F$. Suppose that $\mathbf{b} \overset{m}{\geq} \mathbf{b}'$ and $\mathbf{a}' \overset{m}{\geq} \mathbf{a}$. The following theorem compares the systems when we change from $\mathbf{b}, \mathbf{a}$ to $\mathbf{b}', \mathbf{a}'$.

Theorem 3.7

$$P\{(1,1); (l, \mathbf{b}, m, G), (k, \mathbf{a}, n, F)\} \geq P\{(1,1); (l, \mathbf{b}', m, G), (k, \mathbf{a}', n, F)\}.$$

Proof. This theorem follows from the fact that as we switch from $\mathbf{b}, \mathbf{a}$ to $\mathbf{b}', \mathbf{a}'$ we decrease $Y_{(1)}$ stochastically and increase $X_{(1)}$ stochastically. $\diamond$

Every structure has a dual structure. The properties of a structure can be translated into properties for its dual structure. In El-Neweihi and Sethuraman (1991) we gave examples of this while studying the role of a module in a system. In a similar fashion, one can translate the results above on the role of a group of modules in a system into results for its dual system. Since this can be done in an obvious way following El-Neweihi and Sethuraman (1991) , we will not list these results here.

4. Generalizations to multistate systems.

Consider a component which can be at one of $M+1$ levels of performance $0, 1, \ldots, M$, also called states. State 0 represents total failure and state M stands for perfect functioning, while the other states represent increasing intermediate levels of performance. The nonincreasing right continuous stochastic process $\{X(t), t \geq 0\}$ describes the state of a component at various points in time. Let $T^j = \inf\{t : X(t) \leq j\}$ be the random variable representing the exit time from states $\{j+1, \ldots, M\}$ where $j = 0, \ldots, M-1$. Clearly $\{T^j > t\} = \{X(t) > j\}$. A multistate c-out-of-d system of d multistate components is defined as follows. At any point in time, if $(X_1(t), \ldots, X_d(t))$ represents the states of the d components, then the state of the system is given by the order statistic $X_{(d-c+1)}(t)$. Let T_i^j be the exit time for the ith component from states $\{j+1, \ldots, M\}$ and let $F_i^j(t)$ be its distribution, $i = 1, \ldots, d$. Then the exit time T^j for the system from the states $\{j+1, \ldots, M\}$ will have survival distribution $\bar{F}^j(t)$ given by $\bar{F}^j(t) = h_{c|d}(\bar{F}_1^j(t), \ldots, \bar{F}_d^j(t))$. Consider modules $Q_1, \ldots, Q_l$, and $P_1, \ldots, P_k$ consisting of multistate components. Suppose that P_i is an $a_i + 1$-out-of-n_i subsystem, $i = 1, \ldots, k$ and Q_i is a $b_i + 1$-out-of-m_i subsystem, $i = 1, \ldots, l$.

Let $\{U_{iq}^j, q = 1, \ldots, m_i\}$ be the exit times, from $\{j+1, \ldots, M\}$, of the components in module Q_i, $i = 1, \ldots, l$ and let $\{V_{ip}^j, p = 1, \ldots, n_i\}$ be the exit times, from $\{j+1, \ldots, M\}$, of the components in module P_i, $i = 1, \ldots, k$. Let U_i^j be the exit time, from $\{j+1, \ldots, M\}$, of the module Q_i, $i = 1, \ldots, l$ and let V_i^j be the exit time, from $\{j+1, \ldots, M\}$, of the module P_i, $i = 1, \ldots, k$. Let $U_{(s)}^j$ be the sth order statistic of $U_1^j, \ldots, U_l^j$ and $V_{(r)}^j$ be the rth order statistic of $V_1^j, \ldots, V_k^j$.

We will denote the probability that at least s of the modules $Q_1, \ldots, Q_l$ exit from $\{j+1, \ldots, M\}$ before r of the modules $P_1, \ldots, P_k$ exit from $\{j+1, \ldots, M\}$ by $P^j\{(s, r); (l, \mathbf{b}, \mathbf{m}, \mathbf{G}); (k, \mathbf{a}, \mathbf{n}, \mathbf{F})\}$.

With this notation we can generalize most of the results in Sections 2 and 3 for binary state systems to results on multistate systems. We will confine ourselves to just one such illustration. We will show how to extend Theorem 3.4 to a multistate system in Theorem 4.1 below.

Theorem 4.1 $P^j\{(s,1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,a,\mathbf{n},\mathbf{F})\}$ *is AI in* $(\mathbf{n},\mathbf{F})$.

Proof.

$$
\begin{aligned}
P^j\{(s,1);(l,\mathbf{b},\mathbf{m},\mathbf{G});(k,a,\mathbf{n},\mathbf{F})\} &= \int P(V^j_{(1)} > t)\,dF_{U^j_{(s)}}(t) \\
&= \int \prod_{i=1}^{k} h_{(a+1)|n_i}(\bar{F}^j_i(t))\,dF_{U^j_{(s)}}(t)
\end{aligned}
$$

This integral is AI in in $(\mathbf{n},\mathbf{F})$ since the integrand is AI in $(\mathbf{n},\mathbf{F})$. $\diamond$

This extension and others of the same nature that follow from the results of Sections 2 and 3 have obvious applications to optimal allocation in multistate systems.

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